Hydrodynamics of granular gases with a two-peak distribution

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Vibrating walls, frequently employed to maintain the temperature (i.e., average velocity) in a granular gas, modify the system strongly, rendering it dissimilar to a molecular one in various aspects. As evidenced by microgravity experiments employing a quasi-two-dimensional (quasi-2D) rectangular box and by 2D simulations, the one-peak velocity distribution is split into two, rendering the stress both nonuniform and anisotropic—without a shear flow and in the absence of gravitation. To account for this, granular hydrodynamics (as first proposed by Haff and later derived employing the kinetic theory) is generalized by introducing two additional variables, with one accounting for the distance between the two peaks and a second for the difference between the average velocities along different directions. The hydrodynamic theory thus generalized relates the velocity distribution to the stress, yielding results that agree with experiments and simulations.

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I. INTRODUCTION

That granular gases differ qualitatively from molecular ones is generally acknowledged. In this paper, we focus on the following difference: In a molecular gas, the velocity distribution has only one peak, positioned at $v = 0$. For every particle flying left, there is one flying as quickly right. In granular gases maintained by vibrating walls, the velocity distribution (as emphasized especially and persistently by Evesque [1]) may have two peaks: Hit by a vibrating wall, the grains are injected into the gas at high velocities. They return to the wall's vicinity much more slowly, after numerous lossy collisions, with much of the initial energy absorbed.

If the energy injection equals dissipation, the velocity distribution $f(v)$ is stationary. It then has, closer to the wall, two peaks: one for the (+) particles leaving the wall swiftly, another for the (–) particles returning much more slowly; see Fig. 1. Note that there are many more (–) than (+) particles, with a correspondingly higher peak in $f(v)$, because the total mass current (averaged over a period of the oscillation) vanishes: With $\langle \rho^+ \rangle \langle v^+ \rangle + \langle \rho^- \rangle \langle v^- \rangle = 0$, we necessarily have

$$\langle \rho^+ \rangle < \langle \rho^- \rangle \text{ along with } \langle v^+ \rangle > \langle v^- \rangle. \tag{1}$$

Further away from the wall, the two peaks slowly merge, with the deviation from the Gaussian distribution, a centrally located single peak of the width $\sqrt{T}$, diminishing. (T is the temperature; the particle mass is set to 1.)

Less obvious but closely related is the fact that the stress in granular gas is—*at vanishing shear rates and in the absence of gravitation*—anisotropic and nonuniform. This is shown by two-dimensional (2D) simulations of Herbst *et al.* [2], who employ boundary conditions realistically modeling vibrating walls. In addition to the two-peak structure for $f(v_x)$, they show that the local mean velocities perpendicular and tangential to the vibrating wall, $\langle v_x \rangle$ and $\langle v_y \rangle$, are different, as it takes many collisions before the injected energy along $\hat{x}$ gets fed into that along $\hat{y}$. Possibly as a result of both, the system develops an anisotropic stress, which is also strongly nonuniform—except

the component σ_{xx} which remains constant. The experiments in a vibro-fluidized granular system under microgravity [3] show similar results: two-peak velocity distribution and an anisotropic, nonuniform stress.

These results clearly imply that granular hydrodynamics (GH)—as first proposed by Haff [4] and later derived employing the kinetic theory [5–8]—is no longer adequate. Its stress σ_{ij} contains, in the absence of a shear rate, only a temperature- and density-dependent pressure P. So force balance $\nabla_j \sigma_{ij} = \nabla_j P \delta_{ij} = 0$ requires a uniform and isotropic stress distribution.

Note that the usefulness of repairing GH is neither restricted to systems under microgravity, nor to a boundary layer. The deviation from the Gaussian distribution remains in the presence of gravitation, and it extends deep into the bulk, occupying around 80% of the volume, while the anisotropy permeates throughout the system; cf. Figs. 3 and 4 below that plot the density, energy, and stress nonuniformities resulting from this deviation. Note also that one cannot simply increase the size of the system to render the midregion larger, and reduce the region of deviation to that of a boundary layer, because if the system is too large, the center cools, forms clusters, and is no longer gaseous; see, e.g., [9,10]. If this is to be avoided, the two walls cannot be too far apart. And the inadequacy of GH pertains to the bulk of all wall-excited granular gases.

In [2], Herbst *et al.* went to great lengths to explore the question of whether or not, given the original form of GH, a local description remains possible. They found it is not, although the system they consider (with a ratio of mean free path to system size between 0.01 to 0.1) should be amenable to a local, fluid-dynamic description. More specifically, defining a local function $G(\rho) \equiv P/\rho T$ that is 1 for ideal gases, they plot G for many systems, with the same local parameters (such as the restitution coefficient α) but different global ones (such as the global area fraction or system size). They found no collapse of the curves at $\alpha = 0.5$ and an approximate one at $\alpha = 0.9$, though the latter is really an illusion created by the
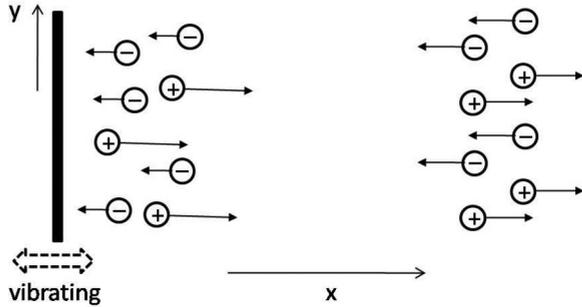
FIG. 1. Perpendicular to the oscillating wall on the left, some $(+)$ particles leave the wall quickly; many more $(-)$ particles return slowly, such that the total mass flow vanishes. Far away from the wall, there are equally many $(+)$ and $(-)$ particles, with the same average velocity. (Arrows depict v_x only.)

semilogarithmic scale; see their Figs. 8 and 9. This lack of a local description is not confined to a boundary layer.

On the other hand, the pressure they employed was $P \equiv \frac{1}{2}\mathrm{tr}(\sigma_{ij})$. Since σ_{xx} and σ_{yy} were observed to have different spatial distributions, one possibility to regain a local description appears to lie in differentiating between them. This is one aspect of what we shall do, and the second is to relate the stress to deviations of the velocity distribution from the Gaussian one. As we shall see, a local description is then indeed restored.

In an early paper [11], Grossman *et al.* studied steady-state properties of inelastic gases in two dimensions, including especially nonequilibrium steady states with asymmetric velocity distributions. They also argued that a hydrodynamic theory should remain relevant even in situations of strong inelasticity, but did not considered anisotropic and nonuniform stresses.

The tool of choice for describing granular gases is, of course, the discrete element method (DEM) calculation, which delivers realistic accounts of many granular phenomena, often quantitatively so; see, e.g., [12–14]. Yet because there is considerable conceptual distance between grain-grain interaction and the macroscopic, collective behavior of granular media, and because much of our heuristic framework for coming to terms with granular media (such as mass and momentum flux) is given by the hydrodynamic theory, it is important to clarify whether GH can be repaired and, if so, how. Moreover, it is important to understand where the anisotropy and nonuniformity of a stationary stress come from and what the connection to the velocity distribution is. At the same time, a given hydrodynamic framework should also facilitate the more ambitious project of setting up a kinetic theory for realistically dense and dissipative grains [15], which would connect micro- and macroscopic properties of granular media directly.

Varying features of granular gases have been observed in the presence of gravitation [16–19], such as how the center of mass moves with the vibration's amplitude and frequency, or the lock-in of a density profile, with the higher density portion either close to or far from the vibrating ground. These complications stem mainly from gravity-induced anisotropy and inhomogeneity, which possibly mask the wall-induced ones. Under microgravity, only the latter exists, making it a simpler problem, easier to understand and account for.

II. ADDITIONAL STATE VARIABLES

Conventional hydrodynamics employs as state variables the densities of mass ρ and momentum ρv, both conserved, in addition to T, which is a measure of the kinetic energy in dilute systems and, more generally, for the average total energy in microscopic degrees of freedom. The pressure is $P = \rho T$ in an ideal gas and, more generally, $p = (\rho \frac{\partial}{\partial \rho} - 1)(w - T s)$, where w and s are the energy and entropy density, respectively. GH has the same variables, though with a modified equation of motion for T, such that it relaxes, going to zero if not replenished. Given its qualitative inadequacy, additional state variables are clearly needed.

To account for a two-peak fluid, it seems necessary to employ as variables two different sets of $T, \rho, \rho v$, for the $(+)$ and $(-)$ particles, along with the pressures P_+ and P_-, as suggested in [20]. Then, perhaps, additional sets of variables for $\hat{y}$ and also $\hat{z}$ are needed. We are afraid this opens a Pandora's box of state variables, neither conserved nor truly independent, rendering the resultant theory arbitrary and unwieldy [21]. It is more sensible, we believe, to retain the conserved variables ρ and ρv, in addition to T as the average kinetic energy of all three directions. It should then suffice, for *a minimal, surgical modification*, to introduce two additional variables.

The first is a measure of the distance between the peaks, which is the crucial second scale of a two-peak distribution. Being a velocity difference, the new variable Δ_i is a vector, odd under time reversal. In our case, only $\Delta_x \neq 0$. (In a single-peak distribution, the width $\sqrt{T}$ is the only scale. But there are many more in a two-peak distribution, which is therefore not well characterized by the two widths $\sqrt{T_+}$ and $\sqrt{T_-}$. Much more characterizing is the pair consisting of the average kinetic energy T and velocity difference Δ_i.)

The second is the difference between T_x and T_y, between the average kinetic energy along $\hat{x}$ and $\hat{y}$. More generally, we have $\delta T_i \equiv T_i - T$, $i = x, y, z$, with $\sum_i \delta T_i = 0$. These are—similar to the order parameter of a nematic liquid crystal [22]—the diagonal elements of a symmetric, traceless tensor. (They do not form a vector because T_x does not distinguish between $\hat{x}$ and $-\hat{x}$.) And this tensor is even under time reversal. In our case, only $\delta T_{xx} = -\delta T_{yy} \neq 0$. However, for reasons to become clear below, we shall, instead of δT_{ij}, employ a slightly different tensor t_{ij}.

III. GRANULAR SOLID HYDRODYNAMICS

Granular solid hydrodynamics (GSH) was derived employing the hydrodynamic procedure [23–25]. Relying on general principles that are valid irrespective of how dense the system is, it leads to equations that include collisions and enduring contacts, which are valid both in the dense, elastoplastic limit and in the rarefied one. For instance, GSH is capable of accounting for the relaxation of T until it is zero.

Although the present system deviates from a rarefied gas in the opposite direction, towards ballisticity, the hydrodynamic procedure still holds if we add variables to characterize this. The reason is the hierarchy of equilibria: Although the two peaks or the kinetic energy in three directions are not in equilibrium with one another, the elements within each are well thermalized.

GSH introduces a granular entropy s_g and the associated temperature T_g, which quantify the energy contained in the fluctuating part of the grains' kinetic and elastic energy. Expanding the energy w in s_g, knowing w has a minimum w_0 for $s_g = 0$, we have

$$w = w_0 + s_g^2/(2b\rho), \quad T_g \equiv \partial w/\partial s_g = s_g/b\rho, \quad (2)$$

with $b = b(\rho)$ positive. Given this energy, s_g relaxes until $s_g \sim T_g = 0$. This is the usual approach for a relaxing quantity, however complex the system. Examples include the order parameter in a Ginzburg-Landau expansion above the phase transition [26]. Only for a rarefied system with negligible elastic contribution may one identify $w = \frac{1}{2}b\rho T_g^2$ with the grains' kinetic energy ρT, implying $\frac{1}{2}bT_g^2 = T$ (for a 2D system). We employ T_g because it is the more general variable, appropriate also for dense media, and because it renders the energetic minimum explicit (in contrast to $w = \rho T$), allowing the standard treatment of T_g as a relaxing quantity.

Taking $b = b_0(1 - \rho/\rho_{cp})^{a_1}$ (with $b_0 > 0$, $a_1 \approx 0.1$, and ρ_{cp} the random closest density) yields, via $P \equiv (\rho\frac{\partial}{\partial\rho} - 1)(w - T_g s_g) = -\frac{1}{2}\rho^2 T_g^2 \partial b/\partial\rho$, the expression

$$P = \frac{a_1\rho\, w}{\rho_{cp} - \rho} = \frac{\frac{1}{2}a_1\rho^2 b\, T_g^2}{\rho_{cp} - \rho}, \quad (3)$$

which is essentially the same expression that is widely used for the kinetic pressure at moderate densities (see [27]), but differs from that at rarefied densities $\sim\rho T$. We could have chosen a more complicated b to interpolate between both expressions, but have refrained from doing so. The reason is first that our purpose in this paper is to generalize GH and create a framework capable of accounting for a stress that is anisotropic in the absence of any macroscopic flows. It is therefore sensible to focus on those aspects in this first step, i.e., the anisotropic stress generated by Δ_i and δT_{ij}, and not the well-known T_g-generated pressure. Second, for lack of sufficient information, we have to be qualitative with the density dependence of the effects presented here. This is the reason c, e, α, β were taken as constant below. As they all appear in the stress with the pressure, it is futile to be too painstaking with the pressure's density dependence. The same holds for the value of ρ_{cp} that is part of the pressure, does not have a generally accepted value, and appears to depend strongly on whether the system is two or three dimensional.

Next, we introduce $\Delta_i, \delta T_{ij}$ as additional variables. Instead of the latter, however, we shall employ t_{ij}, with $T_g^i = T_g + t_{ii}$ for the three principle directions. (Having decided for T_g, it is consistent to use t_{ij}.) As these are also relaxing quantities, we treat them the same way as T_g, postulating a quadratic contribution to the energy,

$$w = w_0 + \left(b\rho T_g^2 + c\rho\Delta_i^2 + e\rho t_{ij}^2\right)/2, \quad (4)$$

with $c, e > 0$. Taking c, e as independent of the density, the pressure remains as given in Eq. (3). (This is assumed for simplicity, as we are more interested in an anisotropic stress and less interested in modifying a given pressure.)

Now the variables Δ_x, t_{xx} are related to the velocity distributions, $f(v_x), f(v_y)$, that are directly measurable quantities; see Fig. 2. (This relation is inessential for the derivation of the hydrodynamic theory. It only serves to visualize the additional
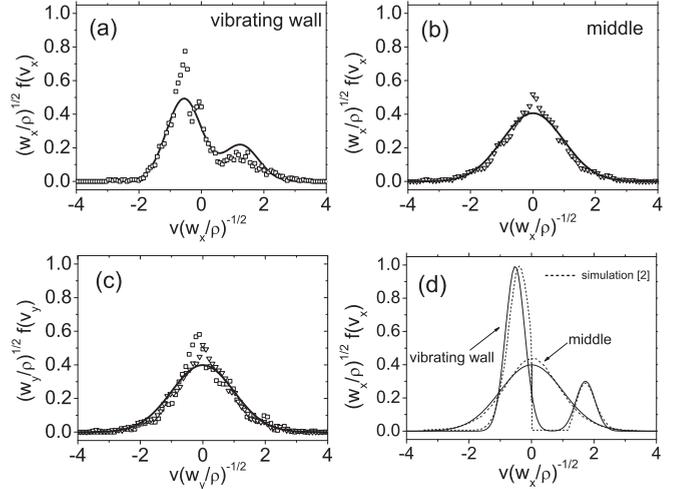

FIG. 2. Velocity distribution as simulated in [2], measured in [3], and parameterized by Eq. (5)—symbols are measurements, dotted lines are the simulation, and full curves show Eq. (5). (a),(b) $f(v_x)$, for the velocity perpendicular to the vibrating walls, with f (a) close to the wall and (b) in the middle. (c) $f(v_y)$. (d) $f(v_x)$. For the two-peak distributions, we have $T_x = 5$ and $1/2$, $\xi = -2.2$ and -1.37, $\alpha = 2.3$ and 3, for (a) and (d), respectively.

state variables.) Denoting the norm as $N \equiv \sqrt{\pi T_x}(1 + \alpha)$, $k_B = \frac{1}{2}$, we take

$$f(v_x) = \frac{1}{N}\left[\alpha\exp\frac{(v_x - \xi)^2}{-2T_x} + \exp\frac{(v_x + \alpha\xi)^2}{-2T_x}\right] \quad (5)$$

and $f(v_y) = f(v_x \to v_y, T_x \to T_y, \xi = 0)$. In comparison to a single-peak distribution that possesses only the width T_x, there are two additional parameters, ξ, α, that account for an asymmetric distribution while ensuring a vanishing mass flow, $\langle v_x \rangle = 0$. (Different widths for the two peaks do not appear necessary; cf. Fig. 2.) Calculating the kinetic energies along $\hat{x}, \hat{y}$, we obtain $w_x = \frac{1}{2}(\rho T_x + \rho\alpha\xi^2)$, $w_y = \frac{1}{2}\rho T_y$. Comparing this to Eq. (4), and pragmatically taking $\alpha = c, \xi = \Delta_x$, we identify

$$T_x = b(T_g + t_{xx})^2/2, \quad T_y = b(T_g - t_{xx})^2/2, \quad (6)$$

to obtain $w = w_x + w_y = \frac{1}{2}\rho b(T_g^2 + t_{xx}^2) + \frac{1}{2}\rho\alpha\xi^2$, which is the same as in Eq. (4), with $b \approx b_0 = e$. [Note that we equate b and e only for the rarefied case, in which w_x, w_y are useful
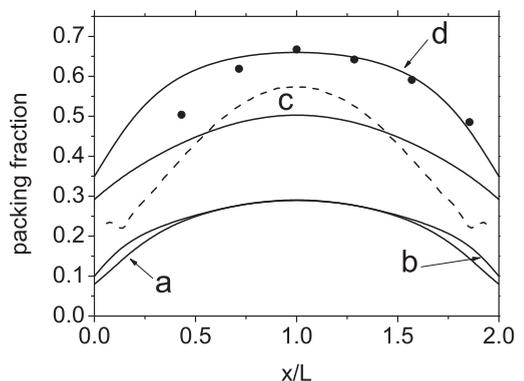

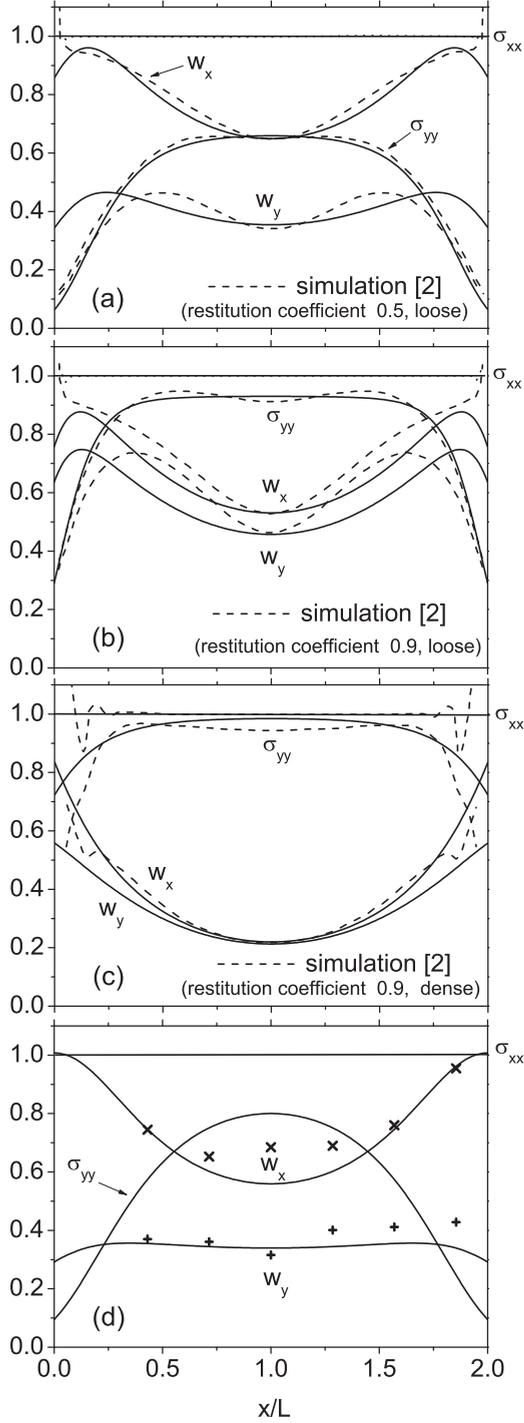FIG. 3. Packing fraction for the same four cases as in Fig. 4.

FIG. 4. Variation of stress and kinetic energy along $\hat{x}$ with σ_{xx} as the unit of stress and energy density. Full curves are hydrodynamic results, taking $w_x = \rho b (T_g + t_{xx})^2/4 + \rho c \Delta_x^2/2$, and $w_y = \rho b (T_g - t_{xx})^2/4$, with $b_0, c, e = 1$, and $\ell_g = 1, 1.1, 2, 1$, $\ell_t = 0.8, 0.9, 1, 0.5$, $\ell_\Delta = 0.18, 0.28, 0.55, 0.3$, $a_1 = 1.38, 1.2, 0.33, 0.9$, $\rho_{cp} = 0.7, 0.7, 0.7, 0.8745$, $-\alpha = 13, 1.49, 0.772, 0.5$, and $-\beta = 3.68, 2.14, 2.18, 3.5$ for (a)–(d), respectively. Symbols are from microgravity measurements of [3] and dotted lines are from simulations of [2].

quantities. At higher densities, b differs significantly from b_0, and the equality no longer holds. In contrast, Eq. (4), obtained by expanding w in t_{ij}, generally holds.]

The same procedure as used for GSH [23] is employed to set up the generalized hydrodynamics. It consists of continuity equations for mass and momentum, $\partial_t \rho = -\nabla_i(\rho v_i)$ and $\partial_t(\rho v_i) = -\nabla_j(\sigma_{ij} + \rho v_i v_j)$, and three balance equations for the relaxing variables s_g, Δ_i and t_{ij},

$$T_g[\partial_t s_g + \nabla_i(s_g v_i - \kappa_g \nabla_i T_g)] = \eta_g v_{ij}^* v_{ij}^* - \gamma_g T_g^2, \quad (7)$$

$$\partial_t \Delta_i + \nabla_j(\Delta_i v_j - \kappa_\Delta \nabla_j \Delta_i) = \alpha \Delta_j v_{ij}^* - \gamma_\Delta \Delta_i, \quad (8)$$

$$\partial_t t_{ij} + \nabla_k(t_{ij} v_k - \kappa_t \nabla_k t_{ij}) = \beta v_{ij}^* - \gamma_t t_{ij}, \quad (9)$$

with $\qquad \sigma_{ij} = P\delta_{ij} - \eta v_{ij}^* - c\rho \alpha \Delta_i \Delta_j - e\rho \beta t_{ij}. \quad (10)$

The first equation is the same as in GSH. It contains a convective ($\sim v_i$), a diffusive ($\sim \kappa_g$), and a relaxative ($\sim \gamma_g$) term, as well as viscous heating, with η_g the viscosity and v_{ij}^* the shear rate. [v_{ij}^* is the traceless part of $v_{ij} \equiv \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$.] With $T_g \sim \sqrt{T}$, one sees that Eq. (7) is also the same as energy balance in GH [15]. Equations (8) and (9) are new, but quite similar to (7). They also each have a convective, diffusive, and relaxative term. Instead of viscous heating, however, there is a linear, off-diagonal Onsager term: with v_{ij}^* as the thermodynamic force and Δ_i as the preferred direction in Eq. (8), and with v_{ij}^* as the force, and no preferred direction, in Eq. (9). α and β are the associated coefficients.

The stress σ_{ij} consists of pressure, viscous stress (with the bulk viscosity neglected), and the two counter Onsager terms, the form of which obey the Onsager reciprocity relation: The thermodynamic forces with respect to β in Eqs. (9) and (10) are v_{ij}^* and $e\rho t_{ij}$, respectively, with different parity under time reversal. Therefore, the Onsager coefficient is β in Eq. (9) and $-\beta$ in Eq. (10). The thermodynamic forces with respect to α in Eqs. (8) and (10) are v_{ij}^* and $c\rho \Delta_i$, respectively, with the same negative parity under time reversal. However, the Onsager coefficients are again antisymmetric, being $\alpha \Delta_i$ in Eq. (8) and $-\alpha \Delta_i$ in Eq. (10), because the preferred direction Δ_i is itself odd under time reversal (same as a magnetic field in the Hall effect). As a result, neither contributes to entropy production. There is no constraint on the sign or magnitude of α, β; both are, in principle, functions of the density. Clearly, σ_{ij} is anisotropic for $\Delta_i, t_{ij} \neq 0$.

IV. VALIDITY OF THE ONSAGER RELATION

As is clear from the above consideration, GH is generalized employing conventional methods of theoretical physics, under the assumption that general principles, especially the Onsager reciprocity relation, remain valid in granular media. As many in the community do not subscribe to it, we explain below why we believe our starting assumption is well justified. First, we distinguish between a *general principle* and an analogy: One has been proven to always hold, hence the name; the other may be good or bad, though no aspect of theoretical physics is imperiled as a result. For instance, there are two versions of the *fluctuation-dissipation theorem* in granular media: one in terms of the true temperature (the one measured with a thermometer), the other in terms of the granular temperature. The former is a general principle that always holds. It is equally applicable to a block of copper and a pile of sand, quantifying,

e.g., how much the volume of each fluctuates. The latter is an analogy—a bad one at times. Similarly, while energy is frequently described as not conserved in granular media, only the kinetic energy is meant. The total energy, including the heat in the grains, is, of course, always conserved.

Then there is the widespread belief that the Onsager relation, another general principle, is not valid in granular media. The argument runs as follows: Since grains collide inelastically and execute irreversible motion, and since the validity of the Onsager relation depends on the *time reversal invariance* of the underlying microscopic dynamics, no reason exists why the Onsager relation may not be violated. Unfortunately, this argument overlooks that the granular kinetic theory (which treats grains as elementary particles) is a mesoscopic, truncated theory, and the apparent lack of time reversal symmetry is a result of this truncation. The microscopic dynamics in sand is, as everywhere else, given by the *reversible* Schrödinger equation for the constituent atoms.

The Onsager relation concerns the pairwise equality (with same or opposite signs) of transport coefficients. Although no one is capable of employing the reversible Schrödinger equation to calculate them, there can be little doubt that the result will comply with the Onsager relation. As this fact may be verified in an appropriate experiment, a proper calculation employing the dissipative kinetic theory must arrive at the same result.

The Onsager relation is valid in all condensed systems, such as elastic medium, liquid crystal, not only molecular gases, because they all share the same microscopic dynamics that possesses time reversal symmetry. Breaking up an elastic medium into little pieces that we call grains cannot change this fact. More fundamentally, that all systems obey *CPT* invariance is a deeply held belief in theoretical physics. In condensed matter, with only electromagnetic interaction, *T* invariance holds alone. The proof of the Onsager relation (such as given in [26]) relies on this completely general *T* invariance, not on the theory one happens to employ or the system one considers.

On the other hand, the violation of the Onsager relation is a result that must not be taken lightly, as it has been obtained by varying authors employing the granular kinetic theory; see, e.g., [28]. There are possibly subtleties that the above argument, though general, is too glib to come to grips with, especially the fact that there is a two-step dissipation in any granular media: macroscopic energy → granular heat → true heat. This fact, however, cuts both ways: Considering only the first step, granular kinetic theory may not be sufficiently complete.

A final settling of the question of whether the Onsager relation holds in granular media is far beyond the scope of the present paper. But we need to stress that this does not change the interpretation given in the next section. The two β in Eqs. (9) and (10) have been set equal on account of the Onsager relation, yet only the second β is needed for calculating the stress, and similarly for α. The first β predicts an anisotropic correction to the granular temperature T_g in the presence of a shear flow v_{ij}^*, which is $t_{ij} = \beta v_{ij}^*/\gamma_t$. This is a very plausible result, as there is no reason why v_{ij}^* in different directions should give rise to the same, strictly isotropic T_g. Measuring both effects at the same time may, in fact, settle the question of Onsager's validity.

V. SOLUTION

Next we solve Eqs. (7)–(9) for the geometry of Fig. 1, with two infinitely extended vibrating walls along $\hat{y}$: one at $x = 0$ and the other at $x = 2L$. First note that setting $v_i, v_{ij}, \Delta_{yy} \equiv 0$, $t_{xx} = -t_{yy}$, and assuming variation only along $\hat{x}$, we have $\sigma_{xy} = 0$ [cf. Eq. (10)], leaving

$$\sigma_{xx} = P - c\rho\alpha\Delta_x^2 - e\rho\beta t_{xx}, \quad \sigma_{yy} = P(x) + e\rho\beta t_{xx}. \quad (11)$$

Force balance $\nabla_j \sigma_{ij} = 0$ requires $\sigma_{xx} = $ const, but leaves σ_{yy} undetermined. Denoting $\ell_g^2 \equiv \kappa_g/2\gamma_g$, $\ell_\Delta^2 \equiv \kappa_\Delta/2\gamma_\Delta$, $\ell_t^2 \equiv \kappa_t 2\gamma_t$, and employing the following boundary conditions:

$$T_g = T_0, \quad \Delta_x = \Delta_0, \quad t_{xx} = t_0 \text{ at } x = 0, \quad (12)$$

$$T_g = T_0, \quad \Delta_x = -\Delta_0, \quad t_{xx} = t_0 \text{ at } x = 2L, \quad (13)$$

the stationary solution, for $\partial_t s_g, \partial_t \Delta_i, \partial_t t_{ij} = 0$, is

$$\frac{T_g}{T_0} = \frac{\cosh \frac{x-L}{\ell_g}}{\cosh \frac{-L}{\ell_g}}, \quad \frac{t_{xx}}{t_0} = \frac{\cosh \frac{x-L}{\ell_t}}{\cosh \frac{-L}{\ell_t}}, \quad \frac{\Delta_x}{\Delta_0} = \frac{\sinh \frac{x-L}{\ell_\Delta}}{\sinh \frac{-L}{\ell_\Delta}}. \quad (14)$$

These results are compared to experiment [3] and simulation [2] in Figs. 3 and 4.

Note that the coefficients $\ell_g, \ell_\Delta, \ell_t$ (generally functions of ρ and T_g) have been taken as constant. Otherwise, obtaining an analytic solution would have been much harder. Similarly, in plotting $w_x, w_y, \sigma_{xx}, \sigma_{yy}$, we take c, e, α, β as constant. These are one reason the agreement is not perfect. On the other hand, improving this seems pointless, as a better agreement, achieved by fitting coefficients to one experiment and one simulation, displaying discrepancies of similar size between themselves, would teach us little beyond the understanding already found.

VI. SUMMARY

The physics of a granular gas has some idiosyncratic features if the granular temperature is maintained by vibrating walls. These are, most notably, first, a two-peak distribution for the velocity perpendicular to the vibrating walls; second, a rather smaller average for the velocity along the walls; and, third, an anisotropic, nonuniform stress. We generalized granular hydrodynamics by introducing two additional variables: Δ_x for the distance between the two peaks, and $t_{xx} = -t_{yy}$ for the difference between the average kinetic energy along $\hat{x}$ and $\hat{y}$. The former is part of a vector, and the latter is part of a symmetric, traceless tensor. Similar to the granular temperature T_g, both variables characterize the velocity distribution of an inelastic gas. All three diffuse and relax, displaying a similar macroscopic behavior. While the scalar T_g gives rise to a hydrodynamic pressure, the vector Δ_i and the tensor t_{ij} contribute to an anisotropic stress. Even under strongly simplifying assumptions, the calculated stress displays qualitative agreement with that of simulations [2] and microgravity experiments [3]. We therefore conclude that granular hydrodynamics as generalized is a useful theory for wall-excited granular gases, and that the basic approach of the hydrodynamic description remains valid.

Finally, a caveat: What we did was obtain the stress by solving the hydrodynamic equations for given boundary values of T_g, Δ_x and t_{xy}; what we did not do is obtain these boundary values as functions of the amplitude and frequency of the

oscillating wall. The latter, of course, is what one controls in an experiment. There are three reasons for not having treated this problem. First of all, any continuum mechanical theory divides into two parts: the bulk structure and the boundary conditions. We have modified the first in this work, but did not consider the second in depth, to which this problem belongs. Second, this is a problem much more easily solved using a DEM simulation than the continuum mechanical theory—it is not an accident that the results of [16] was never reproduced within the framework of GH, though it should be possible. Third, our results differ from that of [2] very close to the walls, until about one granular diameter away. (See the original plot in [2] for this information.) One should not be bothered by this discrepancy because a continuum mechanical theory is expected to fail at this spatial scale. It is probably a result of the short-range conversion of wall-induced motion to the boundary values of the collective variables: T_g, Δ_x and t_{xy}. Clearly, this is not something any continuum mechanical theory is capable of accounting for.

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